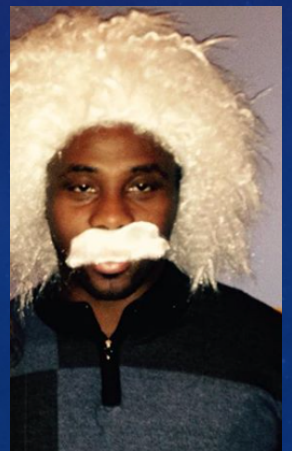
RELATIVITY OF SPACE AND TIME IN POPULAR SCIENCE

RICHARD ANANTUA



COURSE LOGISTICS – SUMMER 2017 A-109 RELATIVITY OF SPACE AND TIME IN POPULAR SCIENCE

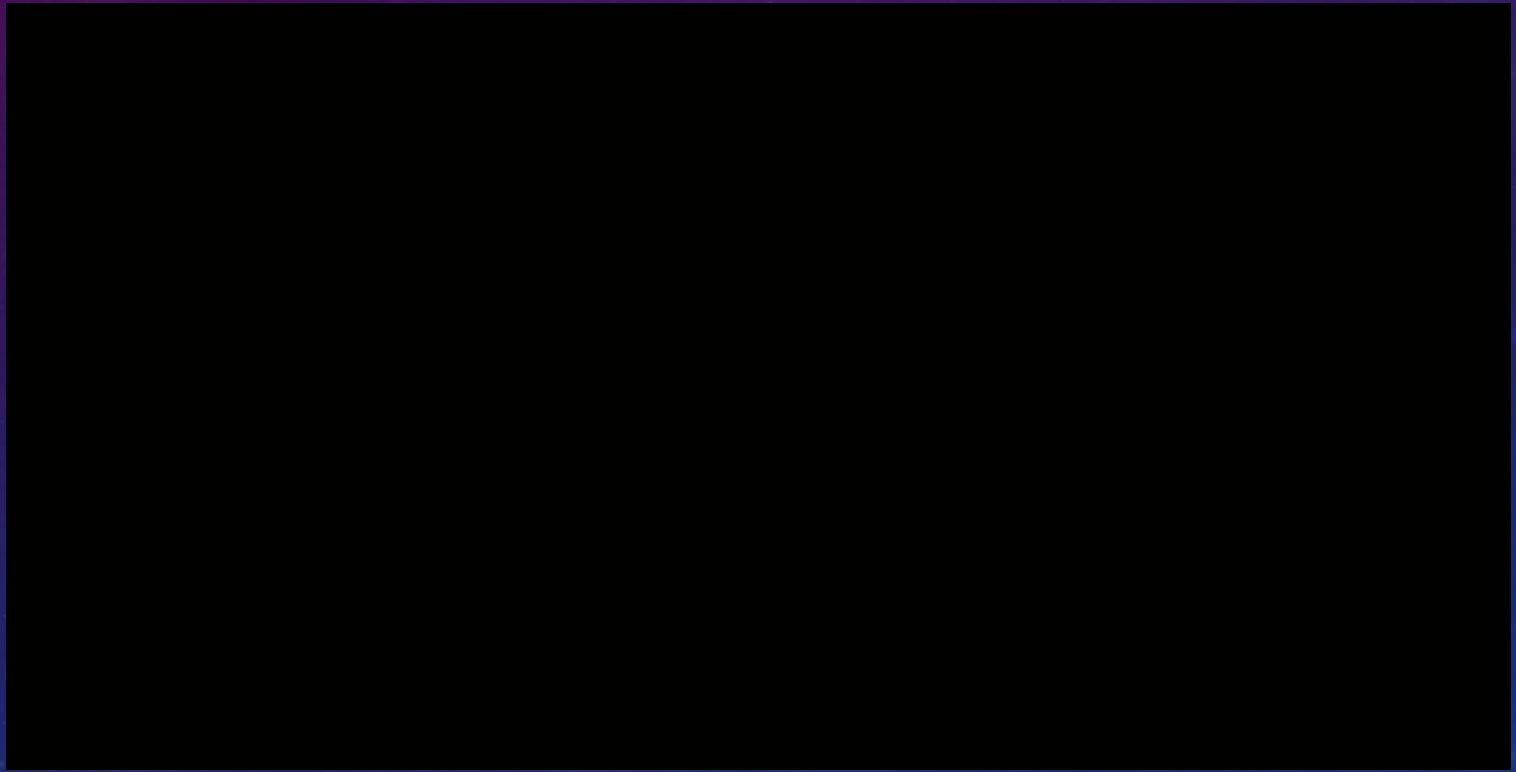
- Instructor: Richard Anantua (ranantua@berkeley.edu)
- Days and time: MWF 1:30-4:00p at Campbell 121 (with 10 min break most classes)
- Office Hours: By appointment
- Materials: *Interstellar*, *Flatland: A Romance of Many Dimensions*, *The Time Machine*, *Predestination*; Optional: Gravity by James Hartle, The Science of Interstellar by Kip Thorne
- Syllabus: http://richardanantua.com/teaching/
- HWs (30%): Weeks 2,3 and 5. LATE POLICY: Up to 1 week late => 50% off; over 1 week late => 100% off
- Midterm Presentation (15%): Week 4
- Midterm Paper (15%): Week 4
- Final Exam (30%): Week 6 (Last week of class)
- Participation (10%): All day, every day (hopefully)

JEOPARDY

Are We Alone?	Time Travel	Astronomy	Astrology	Cosmology or Cosmetology?	To Infinity and Beyond!
100	100	100	100	100	100
200	200	200	200	200	200
300	300	300	300	300	300
400	400	400	400	400	400
500	500	500	500	500	500

JEOPARDY – DAILY DOUBLE

JEOPARDY – FINAL JEOPARDY



MATH METHODS – LINEAR ALGEBRA

- TERMINOLOGY: A *linear* operator A satisfies

$$A(c\mathbf{v}+d\mathbf{w}) = ?$$

for all scalars c,d and vectors **v**,**w**

MATH METHODS – LINEAR ALGEBRA

- TERMINOLOGY: A *linear* operator A satisfies

$$A(c\mathbf{v}+d\mathbf{w}) = cA\mathbf{v}+dA\mathbf{w}$$

for all scalars c,d and vectors **v**,**w**

MATH METHODS – LINEAR ALGEBRA

- Eigenvalues λ and eigenvectors **v** satisfy

$$A\mathbf{v}=\lambda\mathbf{v}$$

where $\mathbf{v} = v_i$, $i = 1,2,\ldots,n$; $A = A_{ij}$, $i,j = 1,2,\ldots,n$ and $\mathbf{v} \neq \mathbf{0}$

- Eigenvalues exists if and only if $(A- \lambda I)\mathbf{v}=0$, which amounts to the matrix $A- \lambda I$ being singular, i.e.,

$$\det(A - \lambda I)=0$$

- TERMINOLOGY: A *positive (negative) definite* operator has all positive (negative) eigenvalues; a *positive (negative) semidefinite* operator has all non-negative (non-positive) eigenvalues; an *indefinite* operator has positive and negative eigenvalues

MATH METHODS – LINEAR ALGEBRA

- Exponential function of one variable as Taylor series

$$e^x = \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \dots$$

where e=2.718281828459045… is the exponential constant

- Matrix exponential

$$e^M = \sum_{n=1}^{\infty} \frac{1}{n!} M^n = I + M + \frac{1}{2!} M^2 + \cdots$$

  - Note Identity matrix satisfies **Iv**=**v** for any vector **v**

MATH METHODS – LINEAR ALGEBRA

- EXERCISE: Find the eigenvalues and eigenvectors of

$$A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \dots & \\ & & & a_{nn} \end{bmatrix} = \mathrm{diag}(a_{11}, a_{22}, \dots, a_{nn})$$

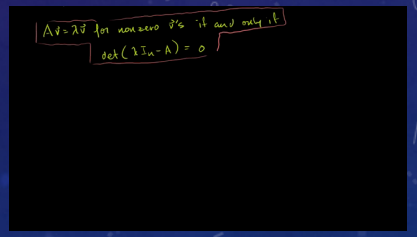
REVIEW OF JUL 3

- Finding determinants

  http://mathworld.wolfram.com/Determinant.html

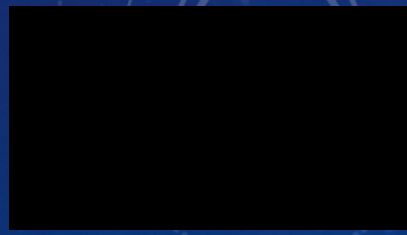
- Finding eigenvalues

  https://www.khanacademy.org/math/linear-algebra/alternate-bases/eigen-everything/v/linear-algebra-example-solving-for-the-eigenvalues-of-a-2x2-matrix



- Finding (families or spaces of) eigenvectors

  https://www.khanacademy.org/math/linear-algebra/alternate-bases/eigen-everything/v/linear-algebra-finding-eigenvectors-and-eigenspaces-example



ICEBREAKER – PHYSICS 2 TRUTHS AND A LIE

- Make 2 true statements and 1 false statement
  - 1 statement must be about yourself
  - 1 statement must be about your home institution
  - 1 statement must be about physics

MATH METHODS – ORDINARY DIFFERENTIAL EQUATIONS

- Differential equations relate functions and their derivatives
- We can directly solve some differential equations by methods such as
  - integration

$$f'(x) = \frac{df}{dx} = \frac{1}{x}$$

  - Separation of variables

$$\frac{dy}{dx} = x^2 y$$

- For many differential equations, the simplest method of solution is making a guess, or *ansatz*

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$$f'(x) = \frac{df}{dx} = \frac{1}{x}$$

$$\Rightarrow f(x) = \ln x + C$$

  - Separation of variables

$$\frac{dy}{dx} = x^2 y$$

$$\Rightarrow \int \frac{dy}{y} = \int x^2\, dx \Rightarrow \ln y = \frac{x^3}{3} + C \Rightarrow y = e^{\frac{x^3}{3}+C}$$

- For many differential equations, the simplest method of solution is making a guess, or *ansatz*

MATH METHODS – ORDINARY DIFFERENTIAL EQUATIONS

- Linear first-order differential equation

$$\frac{dx}{dt} = ax\,(t)$$

Ansatz??

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$$\frac{dx}{dt} = ax\,(t)$$

Ansatz

$$x(t) = e^{at}$$

MATH METHODS – ORDINARY DIFFERENTIAL EQUATIONS

- Linear first-order system

$$\frac{d\vec{x}}{dt} = A\vec{x}(t)$$

Ansatz??

Want to write **x**(t)=e^{At}, but the matrix exponential e^{At} is a matrix while **x**(t) must be a vector

What if we assume A has eigenvectors $\mathbf{E}_j$ satisfying $A\mathbf{E}_j = \lambda_i \mathbf{E}_j$, and write our guess solution as a linear combination of these eigenvectors weighted by $e^{\lambda_i I t}$?

MATH METHODS – ORDINARY DIFFERENTIAL EQUATIONS

- Linear first-order system

$$\frac{d\vec{x}}{dt} = A\vec{x}(t)$$

Ansatz: Assuming A has eigenvalues, i.e., A**v** = λI**v**

$$\vec{x}(t) = \sum_{i=1}^{N} e^{\lambda_i I t} c_i \vec{E}_i = \sum_{i=1}^{N} \sum_{n=0}^{\infty} \frac{(\lambda_i I t)^n}{n!} c_i \vec{E}_i = \sum_{i=1}^{N} \left(\sum_{n=0}^{\infty} \frac{(\lambda_i t)^n}{n!} I^n \right) c_i \vec{E}_i = \sum_{i=1}^{N} \left(I \sum_{n=0}^{\infty} \frac{(\lambda_i t)^n}{n!} \right) c_i \vec{E}_i = \sum_{i=1}^{N} e^{\lambda_i t} c_i \vec{E}_i$$

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$$\vec{x}(t) = \sum_{i=1}^{N} e^{\lambda_i t} c_i \vec{E}_i$$

Solution:

$$\frac{d\vec{x}}{dt} = \frac{d}{dt}\sum_{i=1}^{n} \mathrm{c}_{\mathrm{i}}\vec{\mathrm{E}}_{\mathrm{i}}\, e^{\lambda_i t} = \sum_{i=1}^{n} \mathrm{c}_{\mathrm{i}}\vec{\mathrm{E}}_{\mathrm{i}} \frac{d}{dt} e^{\lambda_i t} = \sum_{i=1}^{n} \mathrm{c}_{\mathrm{i}}(\vec{\mathrm{E}}_{\mathrm{i}}\,\lambda_i) e^{\lambda_i t} = \sum_{i=1}^{n} \mathrm{c}_{\mathrm{i}}(A\vec{\mathrm{E}}_{\mathrm{i}}) e^{\lambda_i t} = A\sum_{i=1}^{n} \mathrm{c}_{\mathrm{i}}\vec{\mathrm{E}}_{\mathrm{i}}\, e^{\lambda_i t} = A\vec{x}$$

MATH METHODS – PARTIAL DIFFERENTIAL EQUATIONS; BOUNDARY CONDITIONS

- Wave PDE:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 f}{\partial t^2}$$

- Ansatz??

MATH METHODS – PARTIAL DIFFERENTIAL EQUATIONS; BOUNDARY CONDITIONS

- Wave PDE:

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 f}{\partial t^2}$$

- Ansatz:

$$f = f_+(x + \mathrm{v}t) + f_-(x - \mathrm{v}t)$$

Works when f_- and f_+ are any differentiable functions!

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But WAIT

- With Periodic Boundary Conditions:

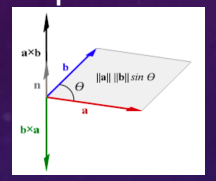
$$f(m\lambda, 0) = A = f(0, nT), m, n \in \mathbb{Z}$$

Solution given by

$$f(x,t) = A\cos(k(x \pm \mathrm{v}t)) = A\cos\left(\frac{2\pi}{\lambda}x \pm \frac{2\pi}{T}t\right)$$

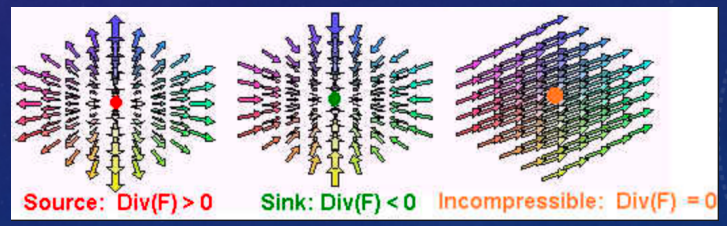
MATH METHODS – MULTIVARIATE CALCULUS

- Vectors in 3D may be multiplied using
  - Dot product: $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = ab\cos\theta$
  - Cross product: $\vec{a} \times \vec{b} = (a_y b_z - b_y a_z)\hat{x} + (a_z b_x - b_z a_x)\hat{y} + (a_x b_y - b_x a_y)\hat{z}$



- A vector field **F**(x,y,z) is specified by assigning a vector to each point in space. At each point we may calculate
  - Divergence (whether the vector field looks like a source or a sink)

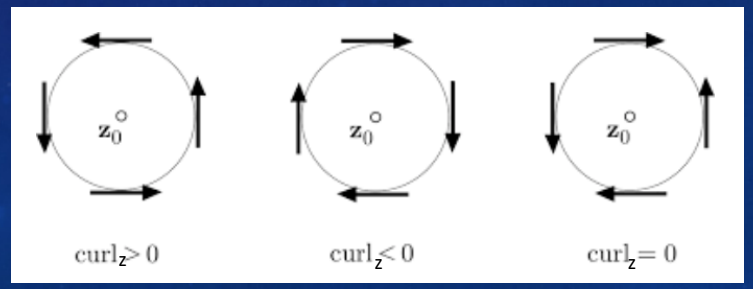
$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$



  - Curl (whether a paddlewheel would rotate in the vector field)

$$\vec{\nabla} \times \vec{F} = \begin{bmatrix} \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \\ \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \\ \frac{\partial F_y}{\partial x} - \frac{\partial F_y}{\partial x} \end{bmatrix}$$

For these xy-plane vector fields, the sign of the z-component of the curl is given



MATH METHODS – MULTIVARIATE CALCULUS

- Gauss' Theorem

$$\oiint_{\partial W} \mathbf{F} \cdot d\mathbf{S} = \oiint_{\partial W} (\mathbf{F} \cdot \mathbf{n}) dS = \iiint_{W} \nabla \cdot \mathbf{F} \, dV$$

The integral of the divergence of a vector field in a volume equals the flux going through its surface

- Green's Theorem

$$\oint_{\partial D} \mathbf{F} \cdot d\mathbf{s} = \iint_{D} (\nabla \times \mathbf{F}) \cdot \mathbf{k} \, dA$$

The integral of the curl of a vector field on a surface D on the xy-plane equals the line integral of the vector field on the boundary of D

- Stokes' Theorem

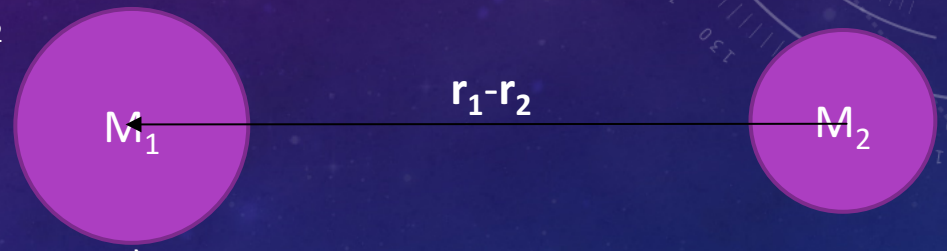
$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \iint_{S} (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

Generalization of Green's theorem to arbitrary surfaces

BEFORE RELATIVITY

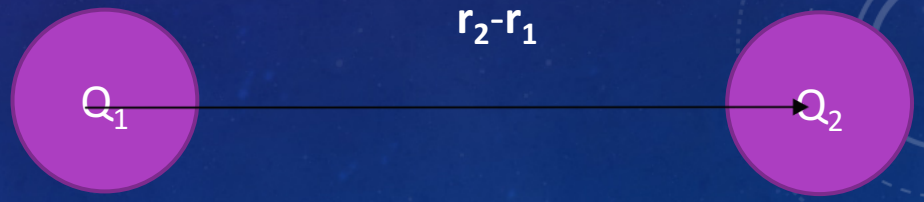
- Classical mechanics – fields and point particles in absolute space and time with “action at a distance”
  - Newton’s law of universal gravitation where G=6.67x10^{-11}Nm^{2}kg^{-2}

$$\vec{F}_{g\ 1\text{ on }2} = G\frac{M_1M_2}{r^2}\frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|}$$



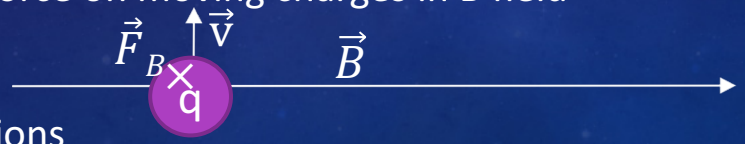
  - Dynamics: Newton’s laws of motion (inertia, $\sum_i \vec{F}_i = m\vec{a}$, action-reaction)
- Electromagnetism – fields and charged point particles
  - Electrostatics: Coulomb’s law with Coulomb constant k=8.99x10^{9}Nm^{2}C^{-2}

$$\vec{F}_{e\ 1\text{ on }2} = k\frac{Q_1Q_2}{r^2}\frac{\vec{r}_2-\vec{r}_1}{|\vec{r}_1-\vec{r}_2|},\ k = \frac{1}{4\pi\varepsilon_0}$$



  - Magnetostatics: Magnetic force on moving charges in B field

$$\vec{F}_B = q\vec{v}\times\vec{B}$$

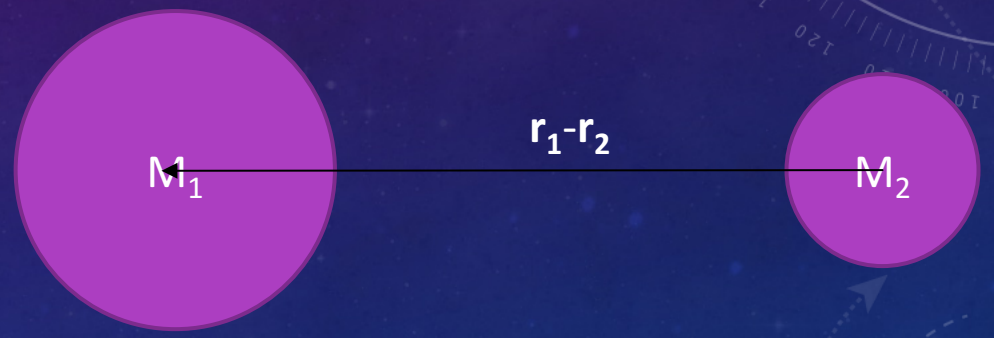


  - Dynamics: Maxwell’s equations

REVIEW OF CLASSICAL MECHANICS – NEWTONIAN GRAVITY

- Newton's Law of Universal Gravitation

$$\vec{F}_{1\text{ on }2} = \frac{GM_1M_2}{r^2}\frac{\vec{r}_1-\vec{r}_2}{|\vec{r}_1-\vec{r}_2|}$$



- Gravitational Potential Energy

$$U_g = -\frac{GM_1M_2}{|\vec{r}_1-\vec{r}_2|}$$

- Kinetic Energy

$$KE = \frac{mv^2}{2} = \frac{p^2}{2m},\ p = mv$$

- Taking M_2 = m to be a unit mass, the gravitational potential (P.E. per unit mass) and field (force per unit mass) due to M_1 = M are:

Gravitational Potential

$$\Phi_g = -\frac{GM}{r}$$

Gravitational Field

$$\vec{g} = -\frac{GM}{r^2}\hat{r}$$

REVIEW OF CLASSICAL MECHANICS – NEWTONIAN GRAVITY

- Potentially useful constants: $M_E=5.98\times10^{24}$kg, $M_S=1.99\times10^{30}$kg, $r_{ES}=1.50\times10^{11}$m, $R_E=6.37\times10^{6}$m, $R_S=6.96\times10^{8}$m, $G=6.67\times10^{-11}$Nm^2/kg^2
- EXERCISE: What is the gravitational potential energy binding the Earth to the Sun?

- EXERCISE: What is the kinetic energy of Earth orbiting the Sun?

REVIEW OF CLASSICAL MECHANICS – NEWTONIAN GRAVITY

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$$U_g = -\frac{6.67\times10^{-11}\,\frac{\mathrm{N\,m^2}}{\mathrm{kg^2}}\cdot 5.98\times10^{24}\mathrm{kg}\cdot 1.99\times10^{30}\mathrm{kg}}{1.50\times10^{11}\mathrm{m}} = -5.29\times10^{33}\mathrm{J}$$

- EXERCISE: What is the kinetic energy of Earth orbiting the Sun?

$$KE = \frac{1}{2}\cdot 5.98\times10^{24}\mathrm{kg}\left(\frac{2\pi\cdot 1.50\times10^{11}\mathrm{m}}{365.25\cdot 24\cdot 60\cdot 60\ \mathrm{s}}\right)^2 = 2.67\times10^{33}\mathrm{J}$$

REVIEW OF ELECTROMAGNETISM – MAXWELL'S EQUATIONS

- Maxwell's equations:

Gauss's Law	$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$
Gauss's Law for Magnetism	$\vec{\nabla} \cdot \vec{B} = 0$
Faraday's Law	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
Ampère Law	$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$

REVIEW OF JUL 5

- Separable differential equations https://www.youtube.com/watch?v=nNHlSB6b1HU
- Linear system of differential equations https://www.youtube.com/watch?v=YUjdyKhWt6E
- Vector analysis:
  - Gradient https://www.youtube.com/watch?v=ynzRyIL2atU
  - Divergence https://www.youtube.com/watch?v=Cxc7ihZWq5o
  - Curl https://www.youtube.com/watch?v=vvzTEbp9lrc&list=PLgivW5IdQExwmTCjK6CizmR4enjICVE-9
- Wave equations https://www.youtube.com/watch?v=NQybjl5mMYk

INTERSTELLAR (FRI JUL 7)