Water Waves.

Phenomena related to water waves are common in daily life:

insect ⟶ boat ⟶ wind ⟶ moon (tidal force)

can all excite water waves.

0) water drop: longer-wavelength travels faster than short-wavelength wave
i). insect, very small length scale, ⟶ surface wave.
(Water strider)
ii). boat wake opening angle is universal. independent of boat. (007 boat chase)
iii) wind excites waves in an elongated river along v_{wind}, wind surfing.
iv) moon tides. (diurnal tides) (show some video here).

Buckingham Π Theorem:

Physical Quantities.

γ (surface tension) $\quad \frac{erg}{cm^2} = \frac{dyn}{cm} = \frac{g\cdot\frac{cm}{s^2}}{cm} = \frac{g}{s^2}$

g (gravity) $\quad \frac{cm}{s^2}$.

$k = \frac{2\pi}{\lambda}$ $\left(\frac{2\pi}{\text{wavelength}}\right)$ $\quad \frac{1}{cm}$

$\omega = \frac{2\pi}{\text{period}}$ $\quad \frac{rad}{s}$ (rad is dimensionless)

ρ $\quad g/cm^3$.

h (water depth) $\quad cm$.

6 quantities − 3 dimensional units = 3 dimensionless groups.

$$\Pi_1 = \frac{\omega^2}{gk} \qquad \Pi_2 = \frac{\gamma k^2}{g\rho} \qquad \Pi_3 = kh$$

relation $\Pi_1 = F(\Pi_2, \Pi_3)$

$$\omega^2 = gk\, F\left(\frac{\gamma k^2}{\rho g},\ kh\right).$$

i) Scale of water strider $\left(\frac{1}{k}\right)$ very small $\sim$ cm

surface tension dominates., so g, h should not appear.

$$\omega^2 = gk \times \frac{\gamma k^2}{\rho g} = \frac{\gamma k^3}{\rho} \quad ① \quad (\text{surface waves}).$$

ii) Boat in a deep river or bay, $k \sim \frac{1}{\text{size of boat}} \gg \frac{1}{h} \Rightarrow kh \gg 1$.

also $\Pi_2 = \frac{\gamma k^2}{\rho g} = \dfrac{70\,\frac{\text{dyn}}{\text{cm}} \times \left(\frac{1}{200\,\text{cm}}\right)^2}{1\,\text{g/cm}^3 \times 10^3\,\frac{\text{cm}}{\text{s}^2}} \ll 1.$

h & γ should drop out.

$$\omega^2 = gk \quad (\text{deep-water wave}) \quad ②$$

iii) Moon tide $\quad k \sim \dfrac{1}{\text{earth radius} \approx 6700\text{km}} \ll \dfrac{1}{h \approx 10\text{km for deepest ocean}}$

again. γ should drop out, but h should stay in.

$$\omega^2 = gk \times \underline{\underline{kh}} = ghk^2 \quad ③ \text{ (shallow water wave)}$$

Why only 1 power of (kh)?

approximate hydrostatic pressure $\quad p = \rho g (h + \xi)$

horizontal equation of motion $\quad \dfrac{\partial v}{\partial t} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = -g\dfrac{\partial \xi}{\partial x}$

mass conservation $\quad \dfrac{\partial}{\partial z}\left(\dfrac{\partial \xi}{\partial t}\right) + \dfrac{\partial v}{\partial x} = 0.$

$\Rightarrow$

$$\omega^2 = ghk^2$$

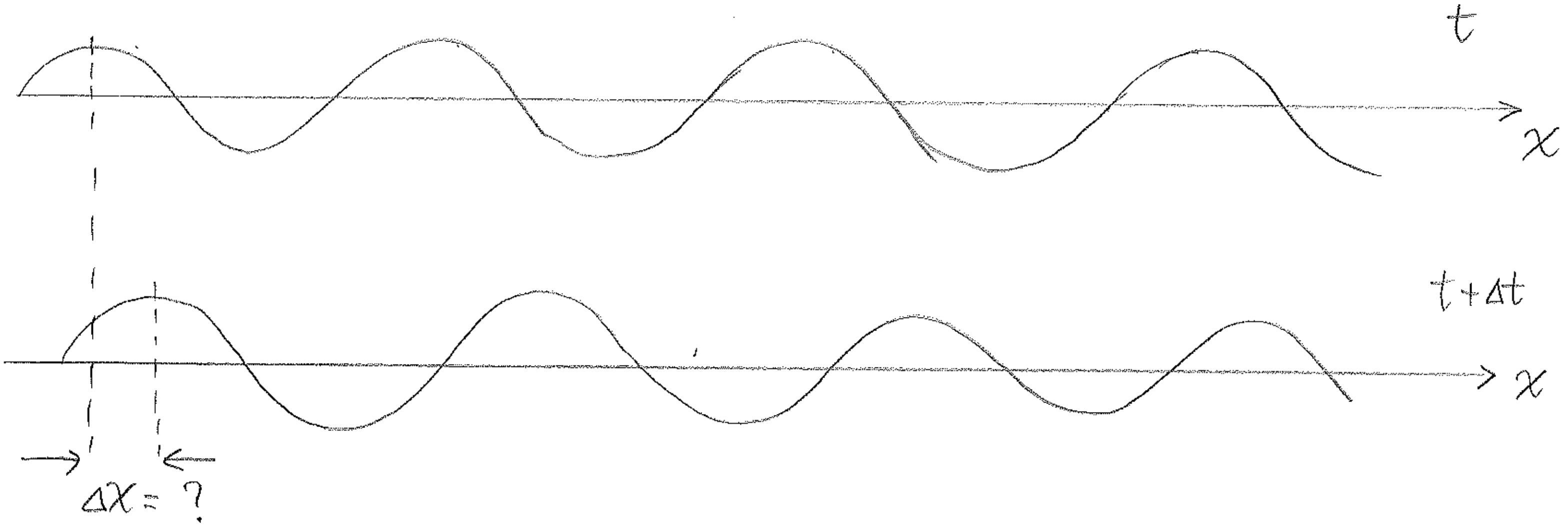
relation between ω & k is called $\boxed{\text{dispersion relation}}$ $\quad \underline{\omega(k)}$

Two velocities describing wave transportation.

phase velocity $\quad v_p = \dfrac{\omega}{k}$

group velocity $\quad v_g = \left(\dfrac{\partial \omega}{\partial k}\right)$

$\sin(xk - \omega t)$



$$\Delta x = \frac{\omega}{k}\Delta t = v_p \Delta t$$

group velocity: velocity of wave package., velocity of transportation for energy, momentum, information, ect.

$v_g < c$ (speed of light)

But v_p can be $> c$., without breaking causality law.

(effect must happen after cause)

$$v_g = \frac{\partial \omega}{\partial k}$$

① $\omega^2 = \frac{\gamma k^3}{\rho} \Rightarrow \omega = \left(\frac{\gamma}{\rho}\right)^{1/2} k^{3/2}$.

$\Rightarrow v_p = \frac{\omega}{k} = \left(\frac{\gamma}{\rho}\right)^{1/2} k^{1/2}$; $v_g = \frac{\partial \omega}{\partial k} = \frac{3}{2}\left(\frac{\gamma}{\rho}\right)^{1/2} k^{1/2}$

surface tension wave: $v_g = \frac{3}{2} v_p > v_p$

longer wavelength, smaller k, slower v_p & v_g.

② deep water wave: $\omega^2 = gk \Rightarrow \omega = (gk)^{1/2}$

$v_p = \frac{\omega}{k} = \left(\frac{g}{k}\right)^{1/2}$; $v_g = \frac{\partial \omega}{\partial k} = \frac{1}{2}\left(\frac{g}{k}\right)^{1/2}$

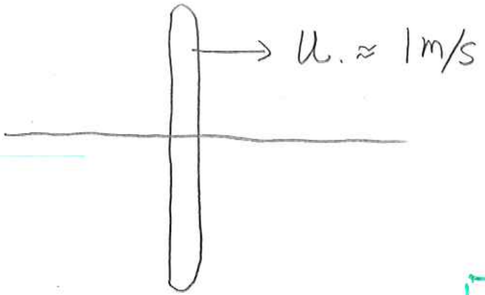
$v_g = \frac{1}{2} v_p < v_p$.

longer wavelength, smaller k, faster v_p & v_g.

③ shallow water wave: $\omega^2 = gk^2 h \Rightarrow \omega = \sqrt{gh}\, k$

$v_p = v_g = \sqrt{gh} \Rightarrow$ shallower water, slower velocity.

↳ both independent of k.



deep water.

$u = v_p$ u determines phase velocity.

would generate two sets of wave

$\boxed{\text{surface tension wave}}$ $u = \left(\frac{\gamma}{\rho}\right)^{1/2} k^{1/2}$

$v_g = \frac{3}{2} u$ > speed of stick.

they are in front of the stick, with

wavelength $\lambda = \frac{2\pi}{k} = 2\pi \left(\frac{\gamma}{\rho}\right) \frac{1}{u^2} = 2\pi \left(\frac{70}{1}\right) \frac{1}{(10^2)^2}$

≈ 0.04 cm   tiny wavelength.

$\boxed{\text{deep water wave}}$ $\left(\frac{g}{k}\right)^{1/2} = u$ ~~(flying rock?)~~

$v_g = \frac{1}{2} v_p = \frac{u}{2} < u$, they lag behind the stick.

$\lambda = \frac{2\pi}{k} = \frac{u^2}{g} 2\pi = 2\pi \times \frac{10^4}{10^3} \approx \underline{60 \text{ cm}}$.

Combine the two : $\omega^2 = \frac{\gamma k^3}{\rho} + gk$. $a+b \geq 2\sqrt{ab}$

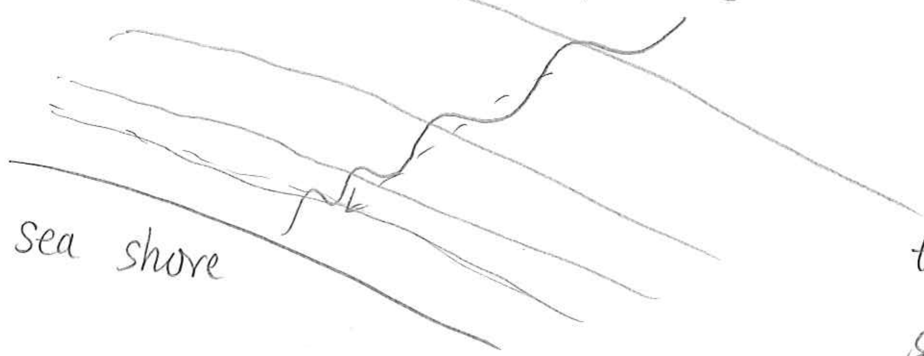
$v_p = \frac{\omega}{k} = \left(\frac{\gamma k}{\rho} + \frac{g}{k}\right)^{1/2} \geq \left(2\sqrt{\frac{\gamma g}{\rho}}\right)^{1/2} \approx \left(2\sqrt{\frac{70 \times 10^3}{1}}\right)^{1/2}$

≈ 23 cm/s. $= v_{p,min}$.

~~if u~~   When $\frac{2\pi}{k} \approx 1.7$ cm if $u < v_{p,\min}$, no wave is gonna be generated.

You can do experiment with it! Suggestion.

Shallow water wave: $v_p = v_g = \sqrt{gh}$



wave fronts get cluster together as they moove close to the shore.

experiment: observe & estimate how water depth (h) increases with distance from the shore.

Suggestion. sloshing of soup

Brain Storm in class:

i) Look back at water drop. video: why do circles ~~move~~ become futher apart from each other. as they mover away from the water drop.

ii) 007 video: wake angle $\Leftarrow v_g \approx \frac{1}{2} v_p$.

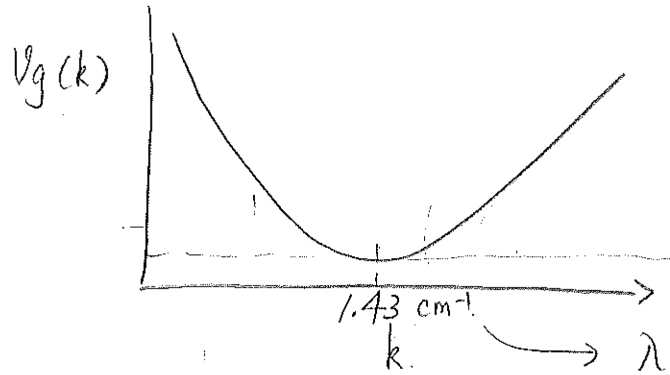
derive it in class $2\theta = 2\arcsin\left(\frac{1}{3}\right)$

$\approx 40^\circ$.

More fun if time allows.

i) $v_g = \dfrac{\partial \omega}{\partial k} = \dfrac{1}{2\omega}\left(\dfrac{3\gamma}{\rho}k^2 + g\right)$

v_g reaches minimum at k ~~when~~ satisfying $\dfrac{\partial v_g}{\partial k} = 0$.

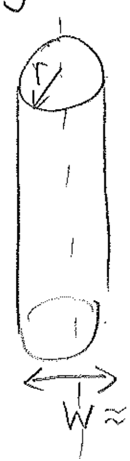


$v_{g,min} \approx 17.7$ cm s^{-1}

$\lambda \approx 4.4$ cm

ii) Diving (video).

$\frac{1}{2}gt^2 = d$
$gt = \left(\frac{2d}{g}\right)^{1/2} g$



$l \approx 1.5$ m.

$W \approx 0.5$ m. $r \approx 0.25$ m.

$\pi r^2 l$ = conserved.

$$\frac{1}{l}\frac{dl}{dt} + 2\frac{1}{r}\frac{dr}{dt} = 0$$

$$\frac{dl}{dt} = v_{human} = gt = g\left(\frac{2d}{g}\right)^{1/2}$$

3-meter diving $d = 3$ m

$$\frac{dl}{dt} \approx v_{human} = (2dg)^{1/2} = (2\times 300 \times 10^3)^{1/2} = (6\times 10^5)^{1/2}$$

$$\approx 774 \text{ cm/s.} \quad \underline{\approx 8 \text{ m/s.}}$$

$$v_{water} = \left|\frac{dr}{dt}\right| = \frac{1}{2}\frac{r}{l}\frac{dl}{dt} = \frac{r}{2l}v_{human} = \frac{0.25}{2\times 1.5}v_{human}.$$

$$= v_{human}/12 \approx 0.66 \text{ m/s.}$$

$v_{water} \lesssim v_g \approx \sqrt{g/k} \approx \sqrt{gr} \approx \sqrt{10^3 \times 25} \approx 50 \text{ cm/s}$

$\boxed{\approx 0.5 \text{ m/s.}}$

Okay, you can avoid splash of water.

iii) Stone a flat piece of rock flying jumping over water
suggest experiment

iv) Sailing boat velocity limit. , faster $\longleftrightarrow$ larger boat

v). What makes it fatal for body jumping into water from cliff?
Swann falls (video).

vi). Sloshing of soup in bowls suggest experiment.

vii) hydraulic jump $E = \frac{1}{2}v^2 + gh$ is conserved

a) in sink
b) tidal bores

$\Big\}$ show video.

Clarification: Group Velocity:

Consider a wave package ~~contai~~ containing two waves at very close wavenumbers.

$$k_1 = k + \Delta k \quad ; \quad \omega_1 = \omega + \Delta\omega = \omega + \frac{\partial \omega}{\partial k}\Delta k.$$

$$k_2 = k - \Delta k \quad ; \quad \omega_2 = \omega - \Delta\omega = \omega - \frac{\partial \omega}{\partial k}\Delta k.$$

$$\sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t)$$

$$= \sin\big((kx - \omega t) + (\Delta k x + \Delta\omega t)\big) + \sin\big[(kx - \omega t) - (\Delta k x + \Delta\omega t)\big]$$

Use $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$

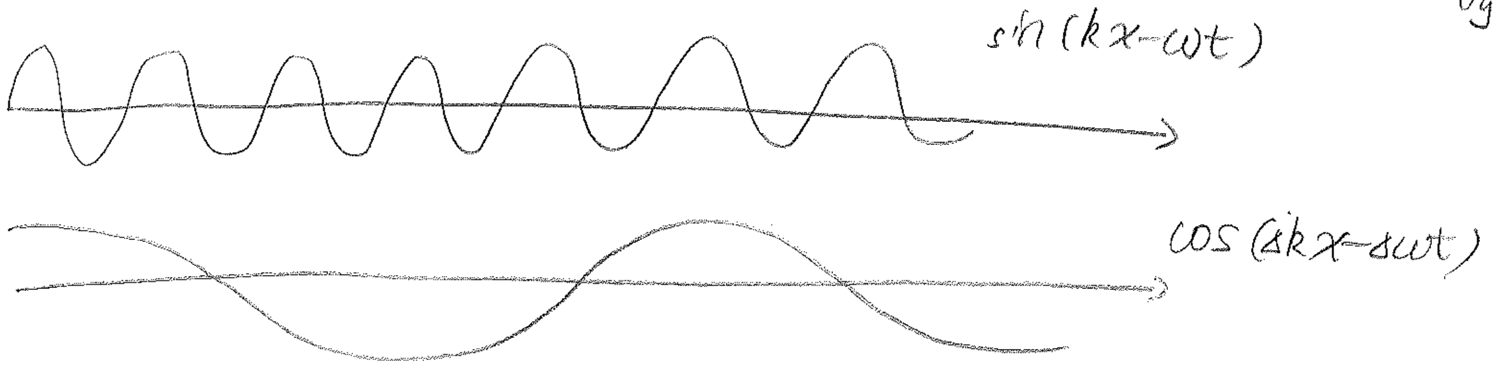
$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$.

α here is $(kx - \omega t)$

β is $(\Delta k x + \Delta\omega t)$

$$= \underline{2\sin(kx - \omega t)}\,\underline{\cos(\Delta k x - \Delta\omega t)}$$

overall shape travels with $\frac{\Delta\omega}{\Delta k} = \frac{\partial\omega}{\partial k} = v_g$.



Water Waves - Continued.

$$\Pi_1 = \frac{\omega^2}{gk} \qquad \Pi_2 = \frac{\gamma k^3}{g\rho} \qquad \Pi_3 = kh$$

$\Pi_1 = F(\Pi_2, \Pi_3)$. (Dispersion Relation)

$\Rightarrow$ Shallow Water : $h \lesssim \lambda = \frac{2\pi}{k}$

in this case, the bottom constrains the water motion, since water just above the bottom must have zero velocity.

Therefore Π_3 must appear in the Dispersion Relation.

For example: ocean wave near the shore.

$\lambda \sim$ meters ; in this case surface tension is negligile compared to gravity (i.e., $\Pi_2 \ll 1$), so surface tension term Π_2 doe not show up.

$\therefore \quad \Pi_1 = \Pi_3 \quad \Rightarrow \quad \boxed{\omega^2 = gk^2 h \quad \text{(shallow water wave dispersion relation)}}$

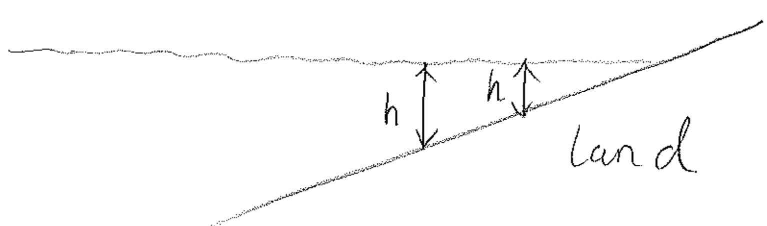
Combine everything:

	Shallow	Deep	Surface
Dispersion Relation	$\omega^2 = gk^2h$	$\omega^2 = gk$	$\omega^2 = \frac{\gamma k^3}{\rho}$
$v_p = \frac{\omega}{k}$	$v_p = \sqrt{gh}$	$v_p = \sqrt{\frac{g}{k}}$	$v_p = \sqrt{\frac{\gamma k}{\rho}}$
$v_g \equiv \frac{\partial \omega}{\partial k}$	$v_g = \sqrt{gh}$	$v_g = \frac{1}{2}\sqrt{\frac{g}{k}}$	$v_g = \frac{3}{2}\sqrt{\frac{\gamma k}{\rho}}$
$v_p \gtrless v_g$?	$v_p = v_g$ (video)	$v_g < v_p$.	$v_g > v_p$
longer λ	v_p (v_g) independent of λ.	longer λ. moves faster	longer λ moves slower.

Observe waves at the shore. (Video)

i) wave breaks.

ii) wave fronts get closer to each other : why ?



h ↓ as the wave approaches to the shore.

In class exercise

Sloshing of soup in a shallow bowl as one walks.

why ?

$10 \times \sqrt{10}$

$$v_p = v_g = \sqrt{gh}$$

$$t = \frac{l}{\sqrt{gh}} = \frac{20\ \text{cm}}{\sqrt{10^3 \frac{\text{cm}}{s^2} \times 2\ \text{cm}}} = \frac{20}{45}\ \text{seconds}$$

$$\approx 0.5\ \text{seconds}$$

natural swing frequency for soup

$$\nu_{soup} \approx \frac{1}{t} \approx 2\ \text{Hz}$$

$$\nu_{walk} \approx \frac{1}{2\pi}\sqrt{\frac{g}{l_{leg}}} = \frac{1}{2\pi} \times \sqrt{\frac{10^3 \frac{\text{cm}}{s^2}}{100\ \text{cm}}} = 0.5\ \text{Hz}$$

$$\because \nu_{soup} \approx \nu_{walk}.$$

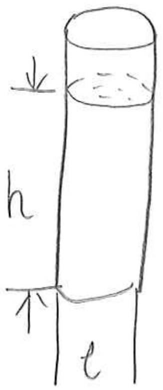
Walking excites the swing of soup resonently.

causing the soup to get out of the shallow bowl.

Experiment: Shallow Bowl v.s. Deep cup.

Walk v.s. mince.

In- Class Exercise



deep cup. $\ell \sim 5$ cm $h \sim 20$ cm.

consider a wave with $k \simeq \frac{1}{\ell}$.

then $kh > 1 \Rightarrow$ Deep water wave.

$$v_p = \sqrt{\frac{g}{k}} \approx \sqrt{g\ell}.$$

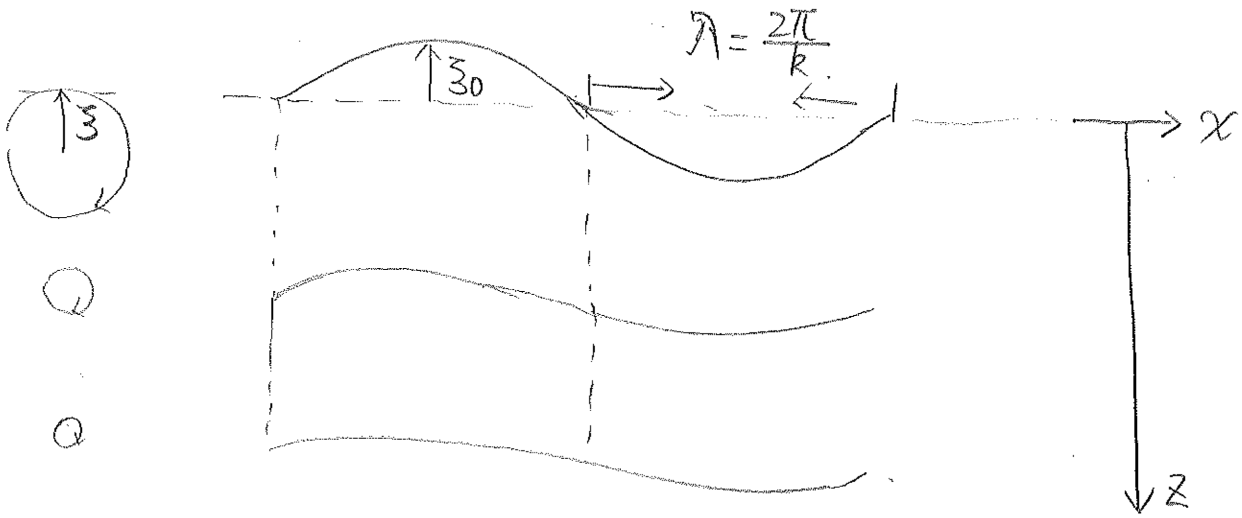
$$t \approx \frac{\ell}{v} \approx \sqrt{\frac{\ell}{g}}$$

$$\nu_{soup} \approx \frac{1}{t} \approx \sqrt{\frac{g}{\ell}} = \sqrt{\frac{10^3 \, cm/s^2}{5cm}} \approx 14 \text{ Hz}$$

$$\nu_{walk} \approx 0.5 \text{ Hz}$$

$\nu_{soup} >> \nu_{walk}$. $\underbrace{\text{, no}}_{\text{no resonant}}$ sloshing.

Why does water wave get weaker as depth z grows?



Physical Picture: $p(z) = \rho g z$.

it's becoming harder & harder to move a mass element at deeper depth z.

incompressible condition is: density does not change $\Rightarrow$

$\vec{\nabla} \cdot \vec{\xi} = 0$ ①

$\rho \frac{\partial \vec{\xi}}{\partial t} = - \vec{\nabla} p' + \rho' g$ ② (equation of motion)

$\vec{\nabla} \cdot$ ② $\Rightarrow$ $\rho \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{\xi}) = - \nabla^2 p' + g \vec{\nabla} \cdot \rho'$ ③

use ① I get $0 = - \nabla^2 p' + g \vec{\nabla} \cdot \rho'$ ④

$\because$ density does not change $\therefore \rho' = 0$

$\therefore$ ④ reduces to $\nabla^2 p' = 0$

Set: $p' = f(z) e^{ikx - i\omega t}$

$\Rightarrow \frac{d^2 f}{dz^2} = k^2 f \Rightarrow f = \underline{G_1 e^{kz}} + G_2 e^{-kz}$.

where G_1 & G_2 are numeric coefficients & $k > 0$

$G_1 \to 0$, otherwise big movement at bottom, which is not allowed.

$\therefore f \propto e^{-kz}$ $\therefore p' \propto e^{-kz + ikx - i\omega t}$

Boundary condition: $\xi_z = 0$ @ $z = h$

② $\Rightarrow$ $-\rho i\omega \vec{\xi} = -\vec{\nabla} p'$ (adopting $\rho' = 0$ & $\vec{\xi} \propto e^{-i\omega t + ikx}$)

$$\Rightarrow \vec{\xi} = \frac{\vec{\nabla} p'}{i\omega\rho} = \frac{ikp'\,\hat{e}_x + \frac{\partial p'}{\partial z}\,\hat{e}_z}{i\omega\rho}$$

$$= \frac{kp'}{\omega\rho}\,\hat{e}_x + \frac{1}{i\omega\rho}\frac{\partial p'}{\partial z}\,\hat{e}_z \Rightarrow \xi_z = \frac{1}{i\omega\rho}\frac{\partial p'}{\partial z}$$

$\xi_z = 0$ @ $z = h$ $\Rightarrow$ $\frac{\partial p'}{\partial z} = 0$ @ $z = h$

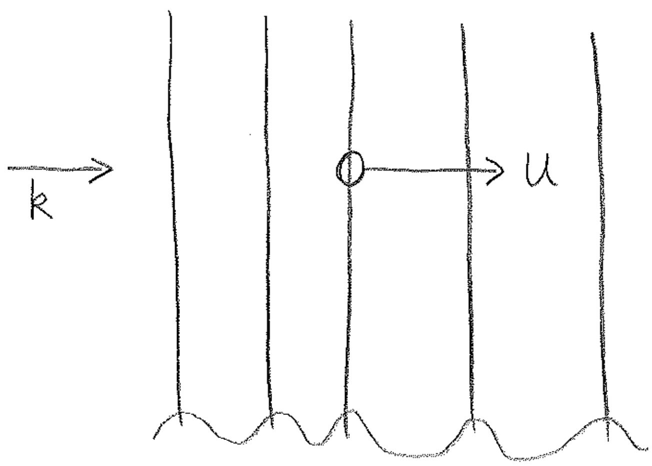
$$\Rightarrow C_1 k e^{kh} - C_2 k e^{-kh} = 0$$

$$\Rightarrow C_1 = C_2 e^{-2kh}$$

i) Deep water: $kh \gg 1$ $\quad C_1 \lll C_2$.

$$p' \approx C_2 e^{-kh + ikx - i\omega t}$$

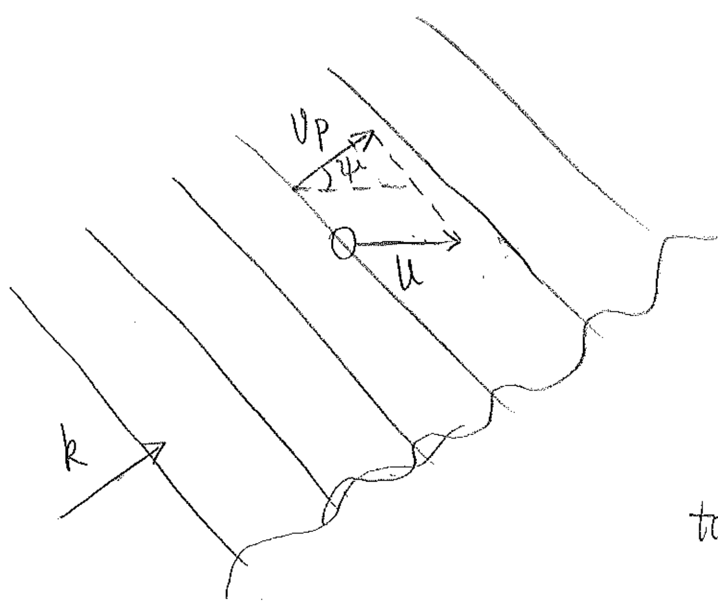
Boat Waves (Revisit).



What's the condition for the boat to stay on a certain wave crest?

i) $\vec{k} \parallel \vec{u}$

$u = v_p$



ii) $u \cos\psi = v_p$

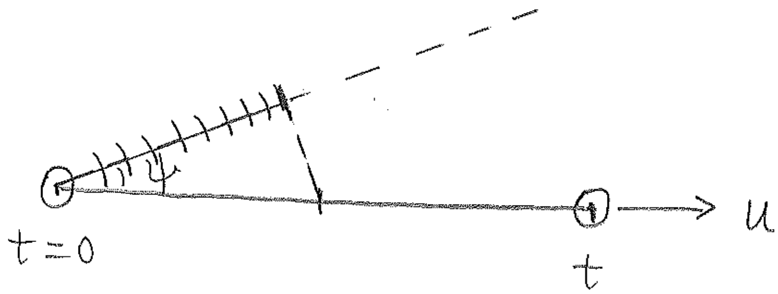
General condition

for the wave front crest

to stay in-phase with the boat

Deep water wave is relevant for boat wave: $v_g = \frac{1}{2} v_p$

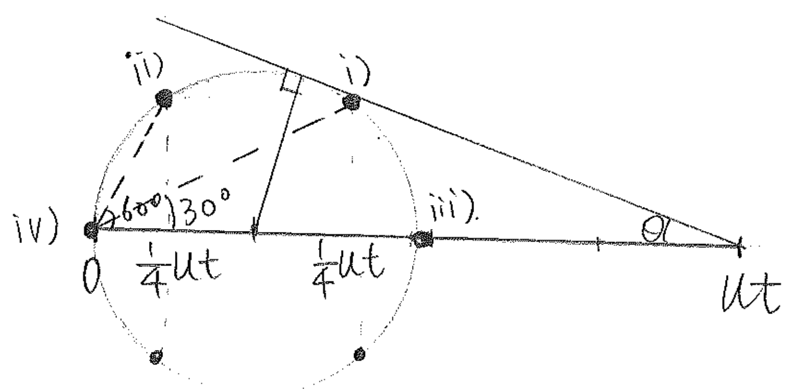
$= \frac{1}{2} u \cos\psi$



Now let's consider several different ψ's.

i) $\psi = 30°$ $\quad v_g = \frac{1}{2} v_p = \frac{1}{2} u\cos(30°) = \frac{u}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4} u$

ii) $\psi = 60°$ $\quad v_g = \frac{1}{2} u \cos(60°) = \frac{1}{4} u.$



iii) $\psi = 0°$ $\quad v_g = \frac{1}{2} u.$

iv) $\psi = 90°$ $\quad v_g = 0$

$\sin\theta = \frac{1}{4} ut / (\frac{3}{4} ut) = \frac{1}{3} \Rightarrow \theta \approx 19°.$

Clarification: in real life, there are usually not any clear visible wave package that one can observe to measure the group velocity directly.

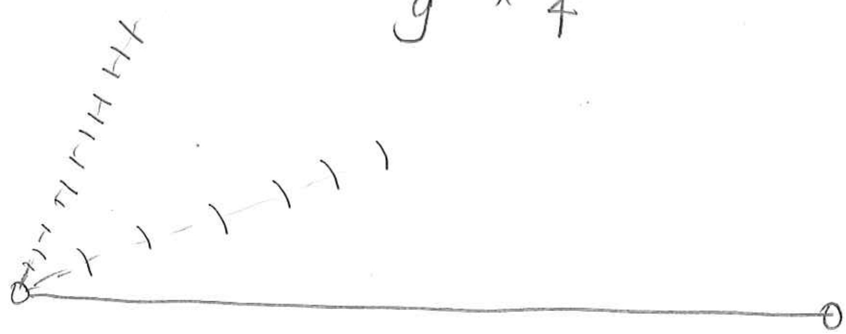
But that does not mean v_g does not exist.

Take boat wave as an example: $\boxed{\text{In-class exercise}}$

$$\left.\begin{aligned} v_p &= u\cos\psi \\ v_p &= \sqrt{g/k} \end{aligned}\right\} \Rightarrow \quad k^{-1} = \frac{(u\cos\psi)^2}{g} \Rightarrow \lambda = \frac{2\pi}{k} = \frac{2\pi u^2}{g}\cos^2\psi$$

i) $\psi = 30°$ $\qquad \lambda_{30°} = \frac{2\pi u^2}{g} \times \frac{3}{4}$

ii) $\psi = 60°$ $\qquad \lambda_{60°} = \frac{2\pi u^2}{g} \times \frac{1}{4}$



Combination of waves of all kinds of λ at various ψ.

But we can try to simplify the case in experiment.